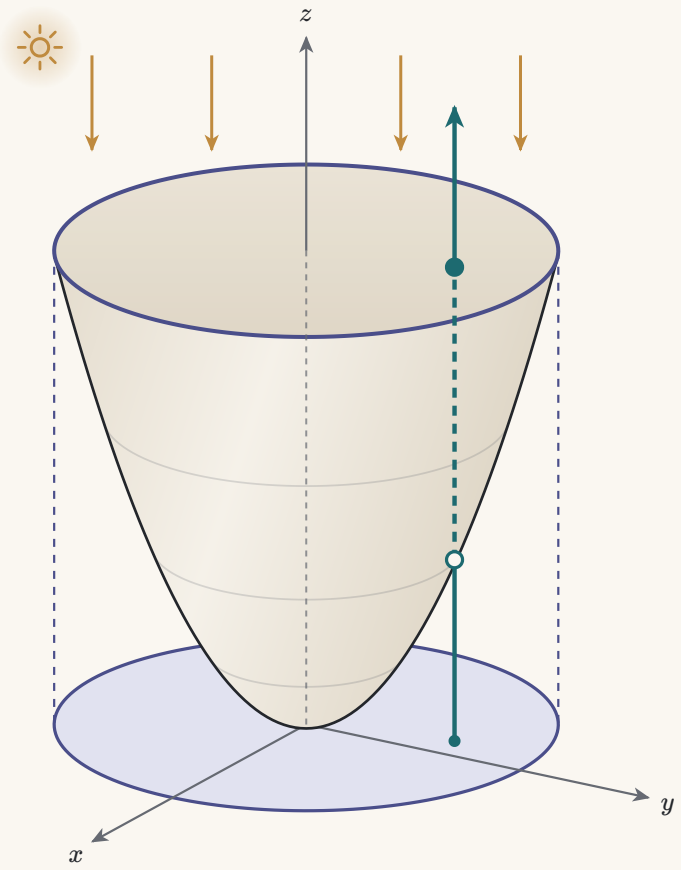
MULTIVARIABLE CALCULUS · A VISUAL EXPLAINER

An Intuitive Way to Define Triple Integrals

Arrows, shadows, and the geometry hiding inside the bounds.

BY OLGA MALYUK



Triple integrals have a reputation. Not because of the integrand — that's usually just some $f(x, y, z)$ minding its own business — but because the bounds look like they were pulled out of a hat. They weren't. Every bound is a piece of geometry, and the next few pages will light the path to them. Almost literally.

01 The general idea

02 Example 1 · A bowl with a lid

03 Example 2 · A wedge, sideways

04 The method on one card

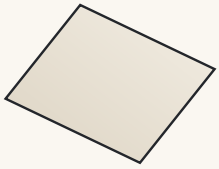
05 What can go wrong?

06 Your turn

BEFORE WE START

Know your surfaces

This method is visual, so it needs one piece of equipment: recognizing a few common surfaces from their equations. If you see $z = x^2 + y^2$ and your brain says *"bowl"*, you're ready. If it says *"... a function?"*, spend ten minutes with these first.

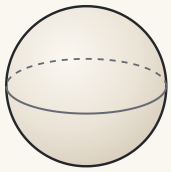


PLANE

$ax + by + cz = d$

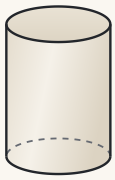
PARABOLOID

$z = x^2 + y^2$



SPHERE

$x^2 + y^2 + z^2 = R^2$



CYLINDER

$x^2 + y^2 = R^2$

SECTION ONE

01 The general idea

Every triple integral over a solid region E starts life looking like this:

$$\iiint_E f(x, y, z)\, dV$$

In setup problems, the integrand is rarely the hard part. The hard part is E: a three-dimensional shape that has to be translated into bounds stacked on three integral signs. So the real question is not *"how do I integrate?"* but *"how do I describe this solid?"*

Reading the order

We begin by choosing an order of integration — say $dz\, dy\, dx$. Read the differentials in the order they appear:

1 — dz

First, move in the **z-direction**.

2 — dy

Then describe how **y** varies.

3 — dx

Finally, describe how **x** varies.

That is also the order in which the nested integrals are handled: from the innermost integral outward.

$$\int \left[\int \left[\underbrace{\int f(x, y, z)\, dz}_{\text{1st}} \right] dy \right] dx$$

The order isn't decoration. It's a set of travel directions through the solid.

Arrows and shadows

1 THE ARROW

Fix the remaining variables — here (x, y) — and move along the first direction of integration: straight up, in z.

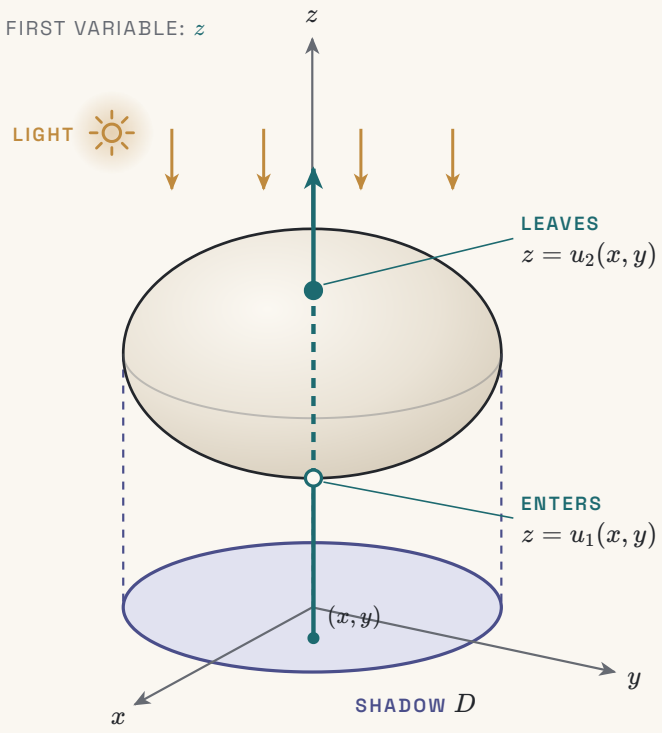
2 ENTER · LEAVE

Where the arrow enters and leaves the solid gives the bounds for the first variable:

$$u_1(x,y) \le z \le u_2(x,y)$$

3 THE LIGHT → THE SHADOW

Shine a light in the same direction. The solid casts a shadow D on the plane of the two remaining variables. Arrows fired from inside D hit the solid; from outside, they miss.



Then repeat, one dimension down

3D		2D		1D
SOLID E	PROJECT →	SHADOW D	PROJECT →	INTERVAL
↓ arrow in z		↓ arrow in y		↓ read it off
$u_1(x,y) \le z \le u_2(x,y)$		$g_1(x) \le y \le g_2(x)$		$a \le x \le b$

Everything slots together:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

SHADOW OF THE SHADOW · INSIDE THE SHADOW · THE ARROW

A useful pattern

The bounds of a variable can only depend on variables we haven't dealt with yet.

For the order $dz\,dy\,dx$, that gives a strict chain of command:

FIRST	dz	$u_1(x,y) \le z \le u_2(x,y)$	may depend on x and y
↓			
SECOND	dy	$g_1(x) \le y \le g_2(x)$	may depend on x
↓			
LAST	dx	$a \le x \le b$	constants only

Why? Each integral uses up its variable. Once the dz-integral is done, z is gone — so the y-bounds have no z to talk about. By the time you reach dx, only numbers are left.

Treat it as a sanity check, not a theorem to memorize: if an outer bound mentions an inner variable, something has gone wrong.

THE RECIPE a preview — we'll use it twice

1. **Choose the order** of integration — it is your itinerary.
2. **Shoot an arrow** in the direction of the first variable.
3. **Find where it enters and leaves** the solid. Those surfaces are the inner bounds.
4. **Turn on the light** and project onto the plane of the variables that remain.
5. **Repeat in the shadow** one dimension down: arrow, enter, leave.
6. **Finish** with the interval for the last variable.

Let's go shoot some arrows.

EXAMPLE 1

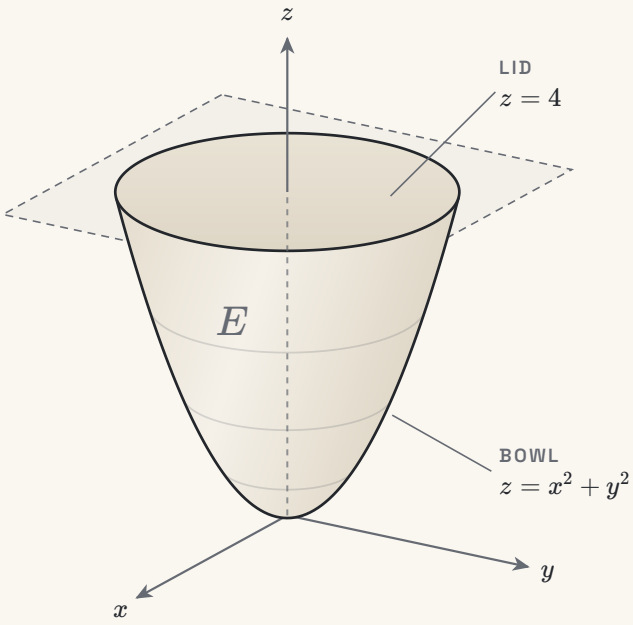
02 A bowl with a lid

PROBLEM Set up $\iiint_E f(x,y,z)\,dV$, where E is the solid bounded by $z = x^2 + y^2$ and $z = 4$, in the order $dz\,dy\,dx$.

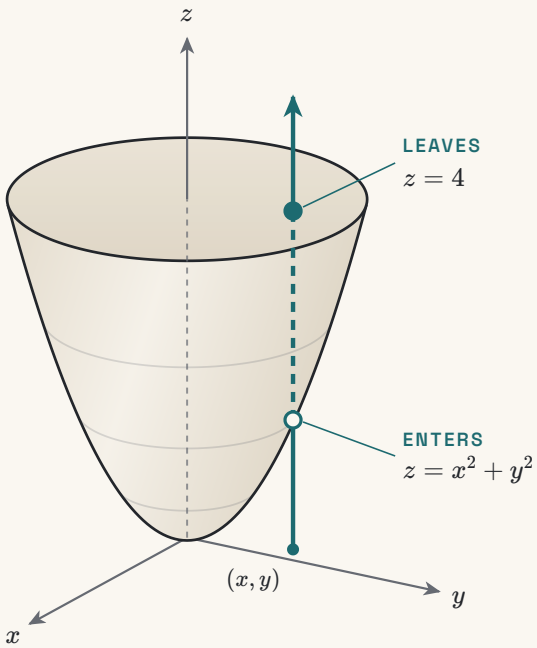
A Sketch the solid

$z = x^2 + y^2$ is a **paraboloid**: a bowl resting on the origin and opening upward. The plane $z = 4$ slices across it horizontally, like a lid.

The solid E lies **above the paraboloid and below the lid** — a smooth, rounded cup of space.



FIRST ARROW · ALONG z



B Shoot the first arrow

The order starts with dz, so the arrow flies in the **z-direction**. Fix x and y and fire upward.

It **enters** through the bowl $z = x^2 + y^2$ and **leaves** through the lid $z = 4$:

Z-BOUNDS

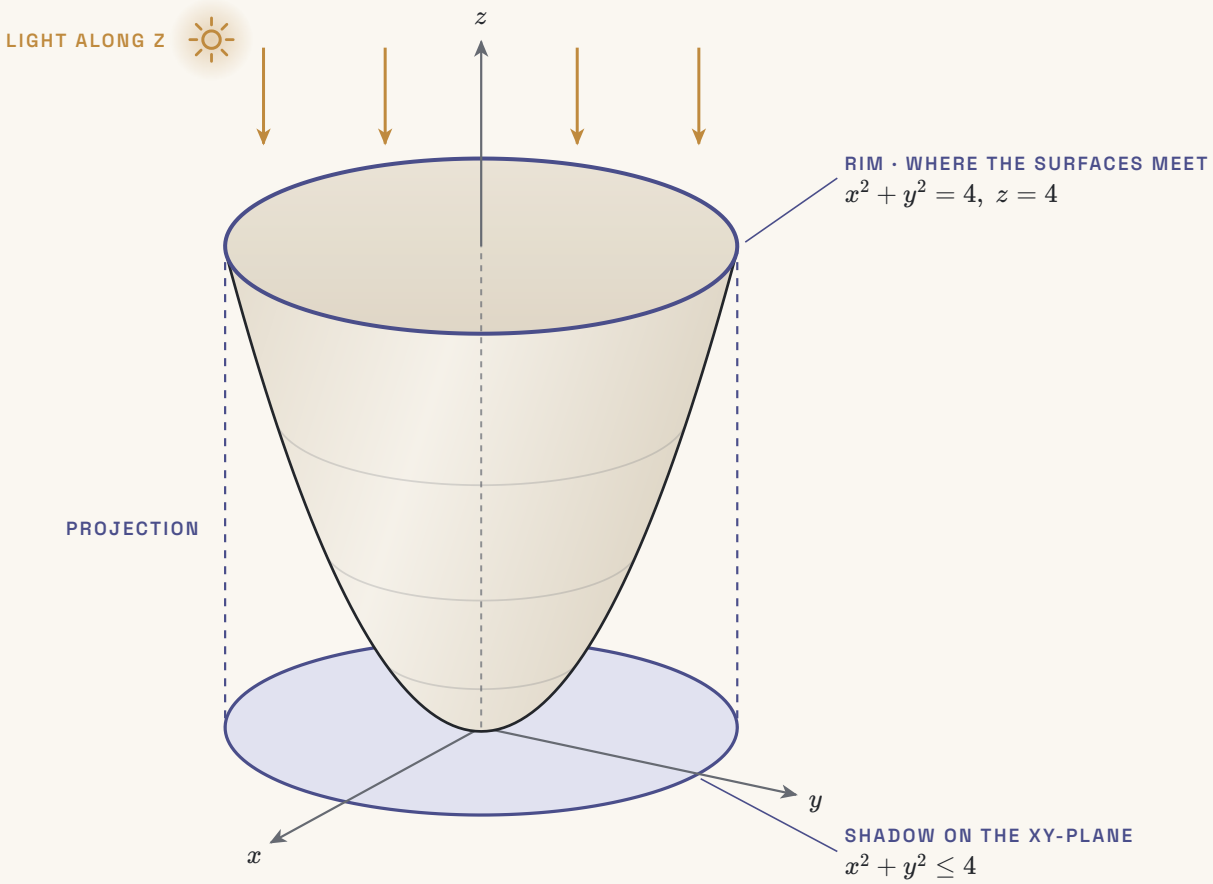
$$x^2 + y^2 \le z \le 4$$

The innermost integral is settled. The other two are still waiting for their bounds:

$$\int_{?}^{?}\int_{?}^{?}\int_{x^2+y^2}^{4} f(x,y,z)\,dz\,dy\,dx$$

C Turn on the light, look at the shadow

Which points (x, y) should we fire from? Switch on a light shining down the z-axis and look at the floor. The shadow of E on the xy-plane is exactly the set of useful starting points.



Here the edge of the shadow is cast by the rim, where the paraboloid meets the lid. Setting the two surfaces equal:

THE RIM

$$x^2 + y^2 = 4$$

So the shadow is the disk of radius 2 centered at the origin:

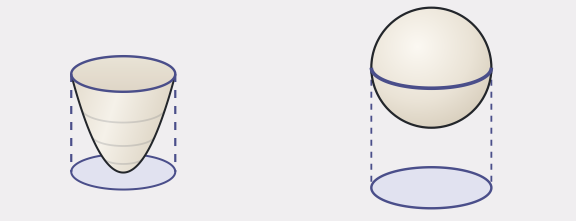
THE SHADOW

$$D = \{(x, y) : x^2 + y^2 \leq 4\}$$

WHERE DOES A SHADOW'S EDGE COME FROM?

The boundary of the shadow comes from the *silhouette* of the solid in the direction of the light. In this case, the silhouette is exactly the rim where the paraboloid meets the plane.

Sometimes the silhouette is an intersection curve; other times it comes from a surface becoming tangent to the viewing direction — like the outline of a ball.



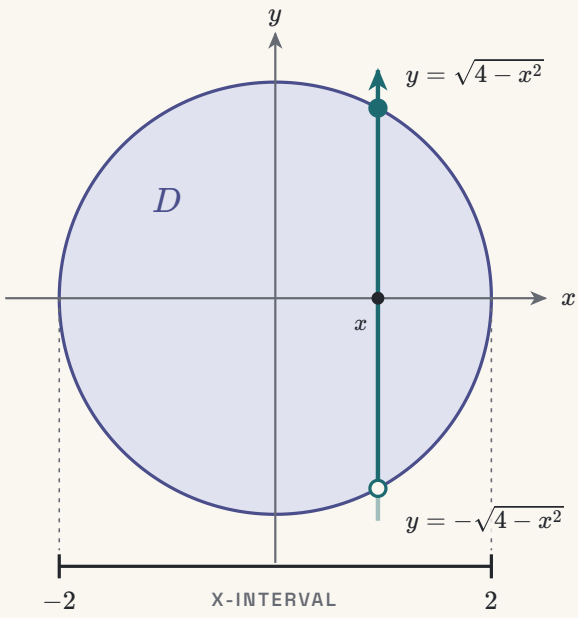
D Find the y-bounds

We're in 2D now, and the next differential is dy. Same trick, one dimension down: fix x and shoot an arrow across the disk in the **y-direction**.

It enters through the lower semicircle and leaves through the upper one. Solving $x^2 + y^2 = 4$ for y:

Y-BOUNDS

$$-\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}$$



E Find the x-bounds

Last variable, last shadow. Squash the disk onto the x-axis:

X-INTERVAL

$$-2 \le x \le 2$$

F Assemble the integral

FINAL SET-UP

$$\iiint_E f\,dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} f(x, y, z)\,dz\,dy\,dx$$

WATCH THE SQUARE

It's $\sqrt{4 - x^2}$, not $\sqrt{4 - x}$. The circle is $x^2 + y^2 = 4$, so the square stays attached to x.

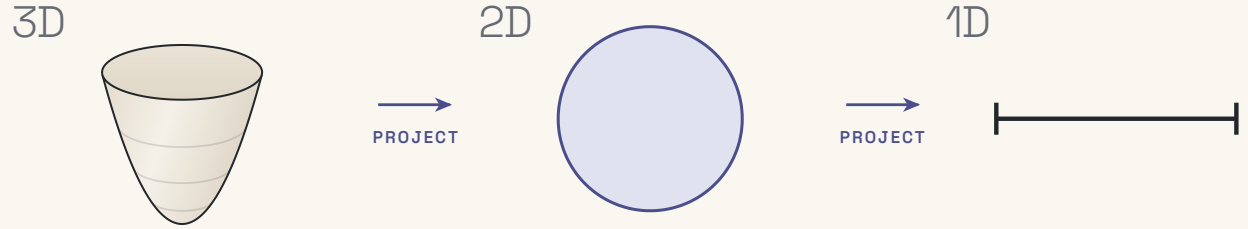
SANITY CHECK

With $f = 1$, the triple integral represents the volume of the solid.

IN HINDSIGHT

What just happened?

We started with a solid, reduced it to a shadow, and reduced the shadow to an interval. **Each integral sign peels away exactly one dimension.**



SOLID

$$x^2 + y^2 \le z \le 4$$

DISK

$$-\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}$$

INTERVAL

$$-2 \le x \le 2$$

EXAMPLE 2

03 A wedge, sideways

PLOT TWIST

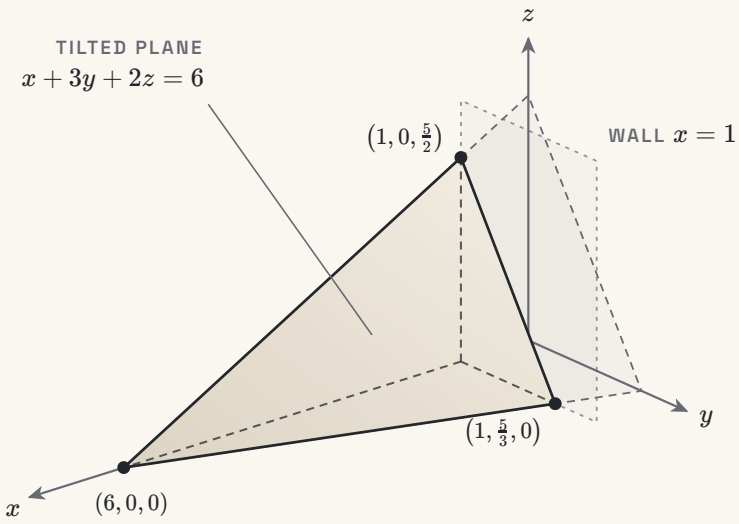
This time the order is $dx\,dy\,dz$. The method doesn't change — only the direction does. The first arrow now flies along the **x-axis**, and the light will shine along x too.

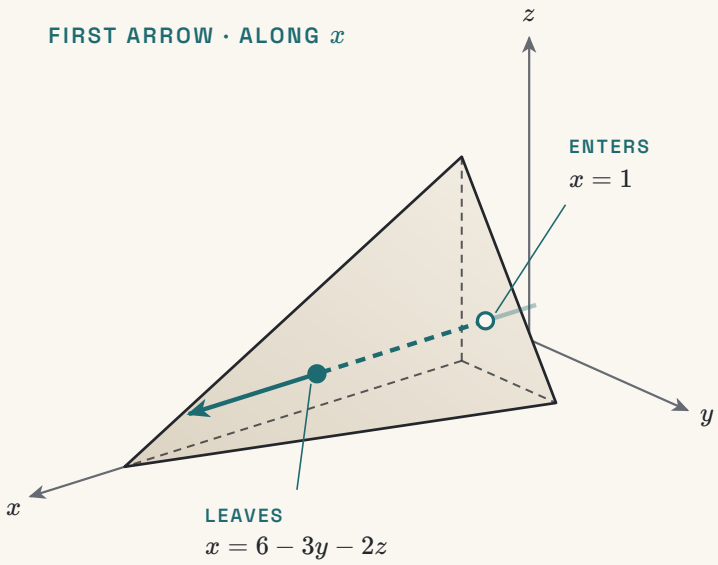
PROBLEM Set up $\iiint_E f(x,y,z)\,dV$, where E is the solid bounded by $x+3y+2z=6$, $z=0$, $y=0$ and $x=1$, in the order $dx\,dy\,dz$.

A Sketch the solid

$x+3y+2z=6$ is a tilted **plane** meeting the axes at $x=6,\ y=2,\ z=3$. The planes $z=0$ and $y=0$ are a floor and a side wall; $x=1$ is a wall parallel to the yz-plane.

The four planes trap a **tetrahedral wedge**, with corners $(1,0,0)$, $(6,0,0)$, $(1,\frac{5}{3},0)$ and $(1,0,\frac{5}{2})$. Only its slanted face points at us; dashed edges are behind it.





B Shoot the first arrow — along x

The first differential is dx, so fix y and z and shoot in the **x-direction**.

The arrow **enters** through the wall $x=1$ and **leaves** through the slanted plane, where $x=6-3y-2z$:

X-BOUNDS

$$1 \le x \le 6-3y-2z$$

The x-bounds depend on y and z — the variables we haven't dealt with yet. Exactly as the pattern promised.

C Turn on the light — project onto the yz-plane

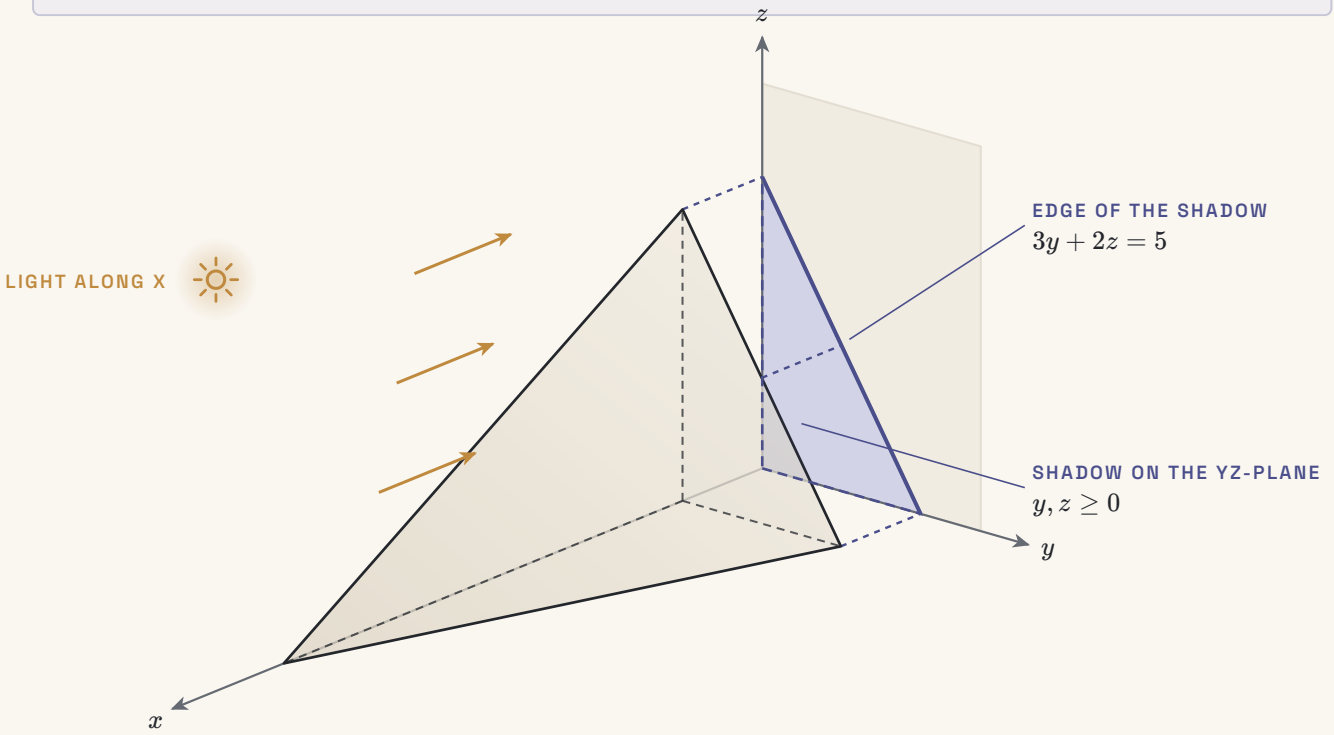
With x handled, shine the light along the x-axis. The shadow now falls on the **yz-plane**.

WHICH PLANE?

The projection plane contains the variables we have left.

x handled → left: y, z → project onto the yz-plane

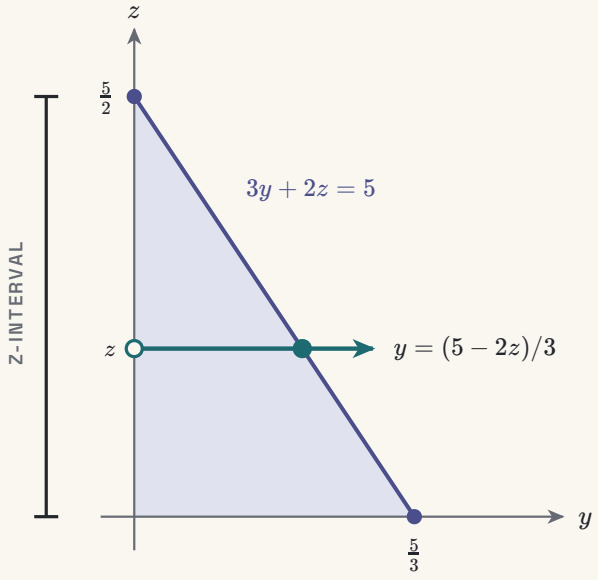
remove z → xy-plane
remove y → xz-plane
remove x → yz-plane



The slanted edge of the shadow is cast where the wall $x = 1$ meets the tilted plane:

$$1 + 3y + 2z = 6 \implies 3y + 2z = 5$$

With $y \geq 0$ and $z \geq 0$, the shadow is a triangle with intercepts $y = \frac{5}{3}$ and $z = \frac{5}{2}$.



D Find the y-bounds

Next differential: dy. Fix z and shoot across the triangle, from the wall $y = 0$ to the slanted edge:

Y-BOUNDS

$$0 \leq y \leq \frac{5 - 2z}{3}$$

E Find the z-bounds

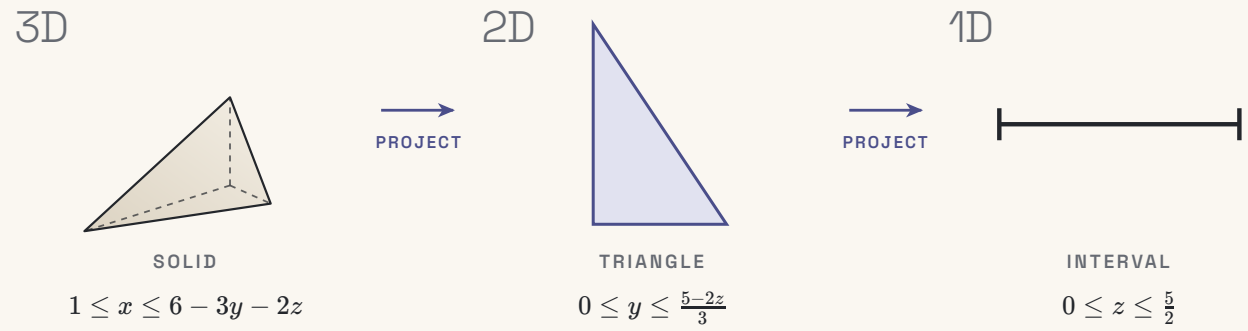
Z-INTERVAL

$$0 \leq z \leq \frac{5}{2}$$

F Assemble the integral

FINAL SET-UP

$$\iiint_E f\,dV = \int_0^{5/2} \int_0^{(5-2z)/3} \int_1^{6-3y-2z} f(x,y,z)\,dx\,dy\,dz$$



Same arrow. Same shadow. Different direction.

	EXAMPLE 1	EXAMPLE 2
ORDER	$dz\,dy\,dx$	$dx\,dy\,dz$
FIRST ARROW	along z	along x
ENTERS / LEAVES	$z = x^2 + y^2$ / $z = 4$	$x = 1$ / $x + 3y + 2z = 6$
SHADOW FALLS ON	xy-plane	yz-plane
SHADOW SHAPE	disk $x^2 + y^2 \le 4$	triangle $3y + 2z \le 5,\ y, z \ge 0$
LAST INTERVAL	$-2 \le x \le 2$	$0 \le z \le \frac{5}{2}$

TAKEAWAY

04 The method on one card

How to build the bounds of a triple integral

1. Pick an **order of integration**.
2. Move in the direction of the **first variable**.
3. Find the surfaces where you **enter and leave** the solid.
4. **Project** the solid onto the plane of the remaining variables.
5. Determine the bounds **in that projection**.
6. Finish with the **interval** for the last variable.

DEPENDENCE Each new pair of bounds may depend only on the variables that remain.

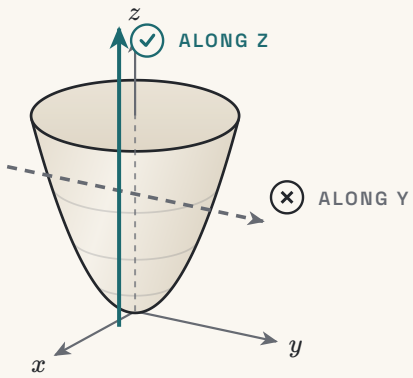
3D solid $\xrightarrow{\text{arrow}}$ 2D shadow $\xrightarrow{\text{arrow}}$ 1D interval

Before you try one on your own: three ways this can go wrong.

DETOURS

05 What can go wrong?

Three classic ways to get lost. Recognizing them is most of the work of avoiding them.



MISTAKE 1

Wrong direction

The order is $dz\,dy\,dx$, but the arrow was fired along y — so the "bounds" it finds belong to a different order entirely.

> **The first differential is dz, so start in the z-direction.**

$$\int_0^{\boxed{y}} \int_0^{x} \int_0^{x+y} f(x,y,z)\,dz\,dy\,dx$$

⊗ the outer x-bound mentions y

MISTAKE 2

Impossible dependence

By the time you integrate in x, the y-integral has already been done — there is no y left. The final answer would still contain a variable that shouldn't exist.

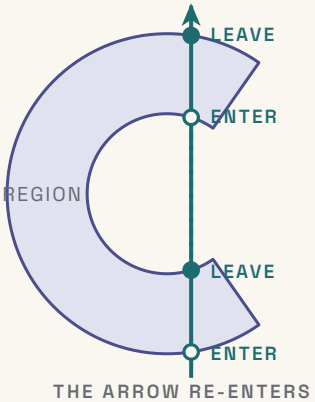
> **Once x is the final variable, its bounds cannot depend on y.**

MISTAKE 3

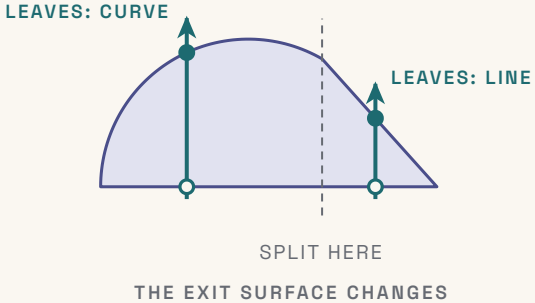
One arrow is not always enough

Two things can happen as you slide the arrow around the shadow. It may **enter and leave more than once** — so no single pair of bounds covers it. Or it may keep entering once, but **through a different surface**, so the formula for the bounds changes halfway across.

> **Either way, one iterated integral is not enough: split the shadow into pieces, write one integral per piece and add them — or try a different order.**



THE ARROW RE-ENTERS

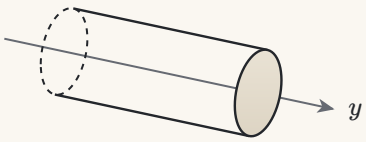


THE EXIT SURFACE CHANGES

YOUR TURN

06 No training wheels this time

PROBLEM Let E be the solid lying **inside both** the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + z^2 = 1$. Set up $\iiint_E f(x, y, z)\, dV$ in the order $dy\, dz\, dx$.



Reading the cylinder. The equation $x^2 + z^2 = 1$ contains no y, so y is free to be anything: this cylinder runs along the y-**axis**, not the z-axis.

1 **Sketch it.** Start where we always start — and notice which surface closes off each end.

2 **Which way does the arrow go?** The order is $dy\, dz\, dx$, so the first arrow travels along...

- ◯ **A** x-direction
- ◯ **B** y-direction
- ◯ **C** z-direction

3 **Enter · leave.** Fix x and z and shoot the arrow through the solid. Where does it enter, and where does it leave? Write the y-bounds.

4 **Turn on the light.** Shine it in the same direction as the arrow and picture the shadow of E on the plane of the variables that remain.

5 **Describe the shadow.** Write the inequalities that define it.

6 **Second arrow.** The next differential is dz. For a fixed x, where does a z-arrow enter and leave the shadow?

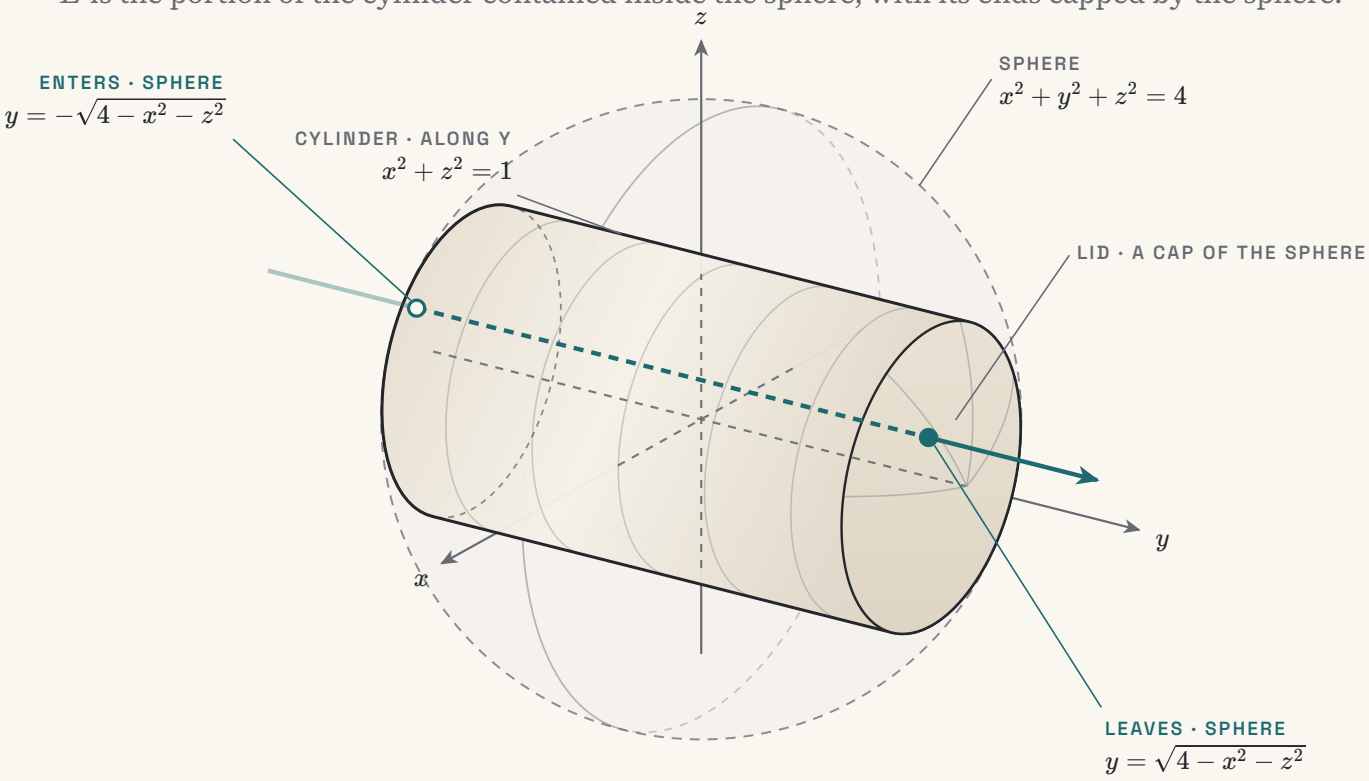
7 **Last shadow.** Only x is left. What interval does the shadow cast on the x-axis?

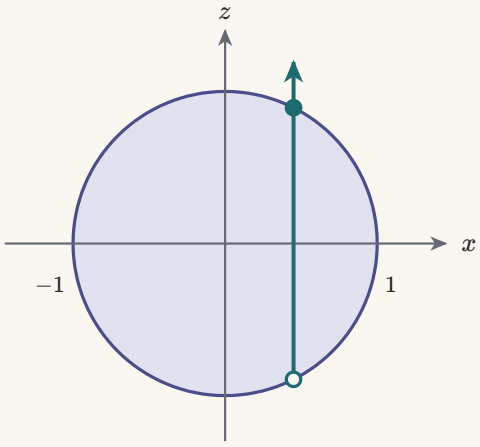
8 **Assemble it.** Write the iterated integral in the order $dy\, dz\, dx$.

Answers on the next page. No peeking until you've committed to something.

Check your work

E is the portion of the cylinder contained inside the sphere, with its ends capped by the sphere.





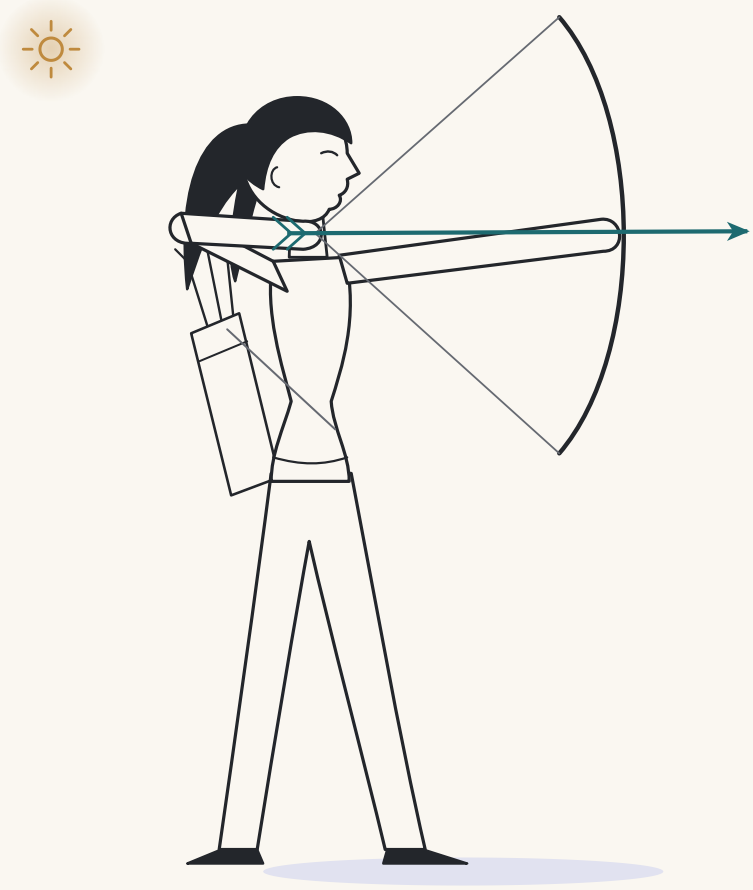
SHADOW ON THE XZ-PLANE

SPHERE → y-bounds

CYLINDER'S SHADOW → z- and x-bounds

FINAL SET-UP

$$\iiint_E f\,dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{4-x^2-z^2}}^{\sqrt{4-x^2-z^2}} f(x,y,z)\,dy\,dz\,dx$$



I think you're ready!

Now go shoot some arrows and slay those integrals.

Just don't forget to turn on the light ;)