PART 1 · THE SETUP

Conditional *Probability*.

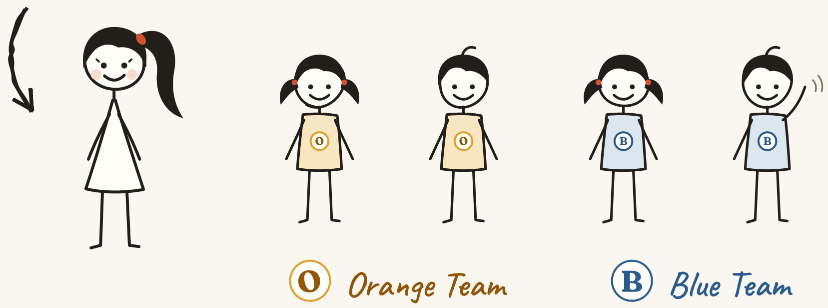
Apparently, knowing things changes things.

A class of **22 students** was divided at random into two teams for an activity. The students were also asked whether they *love* or *hate* math, and the results were as follows:

	LOVE MATH	HATE MATH	TOTAL
O *Orange Team*	9	3	12
B *Blue Team*	4	6	10
Total	13	9	22

Meet the cast

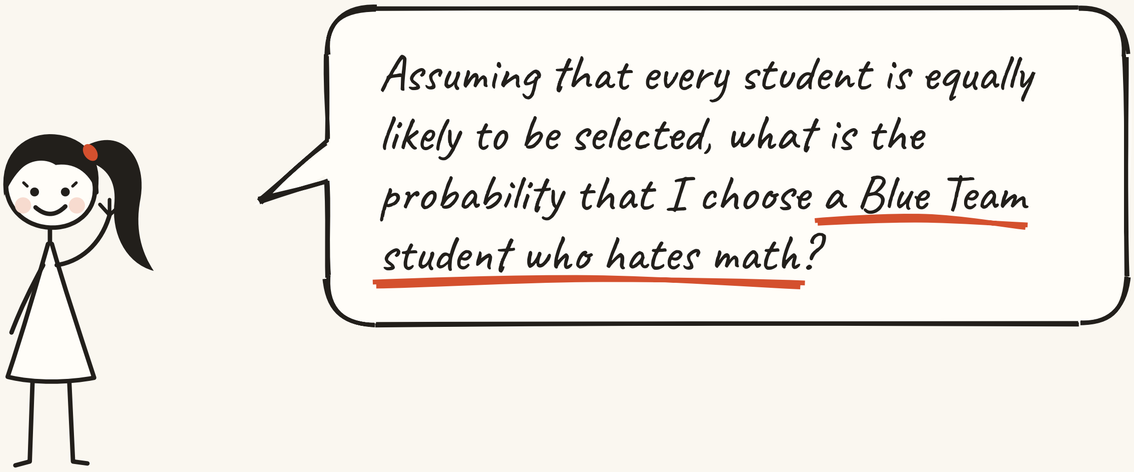
Keep an eye on this table. It's about to get interesting.



PART 1 · THE SETUP

An easy one first.

The teacher asks the class:



Can you guess the answer?

(Go on, have a guess. We'll wait.)

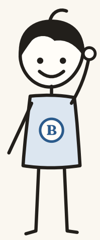
There are **6 Blue Team students who hate math** and **22 students in total.**

	LOVE	HATE	TOTAL
Ⓞ *Orange Team*	9	3	12
Ⓑ *Blue Team*	4	6	10
Total	13	9	22

6 Blue Team students who hate math

$$\frac{6}{22} \approx 27\%$$

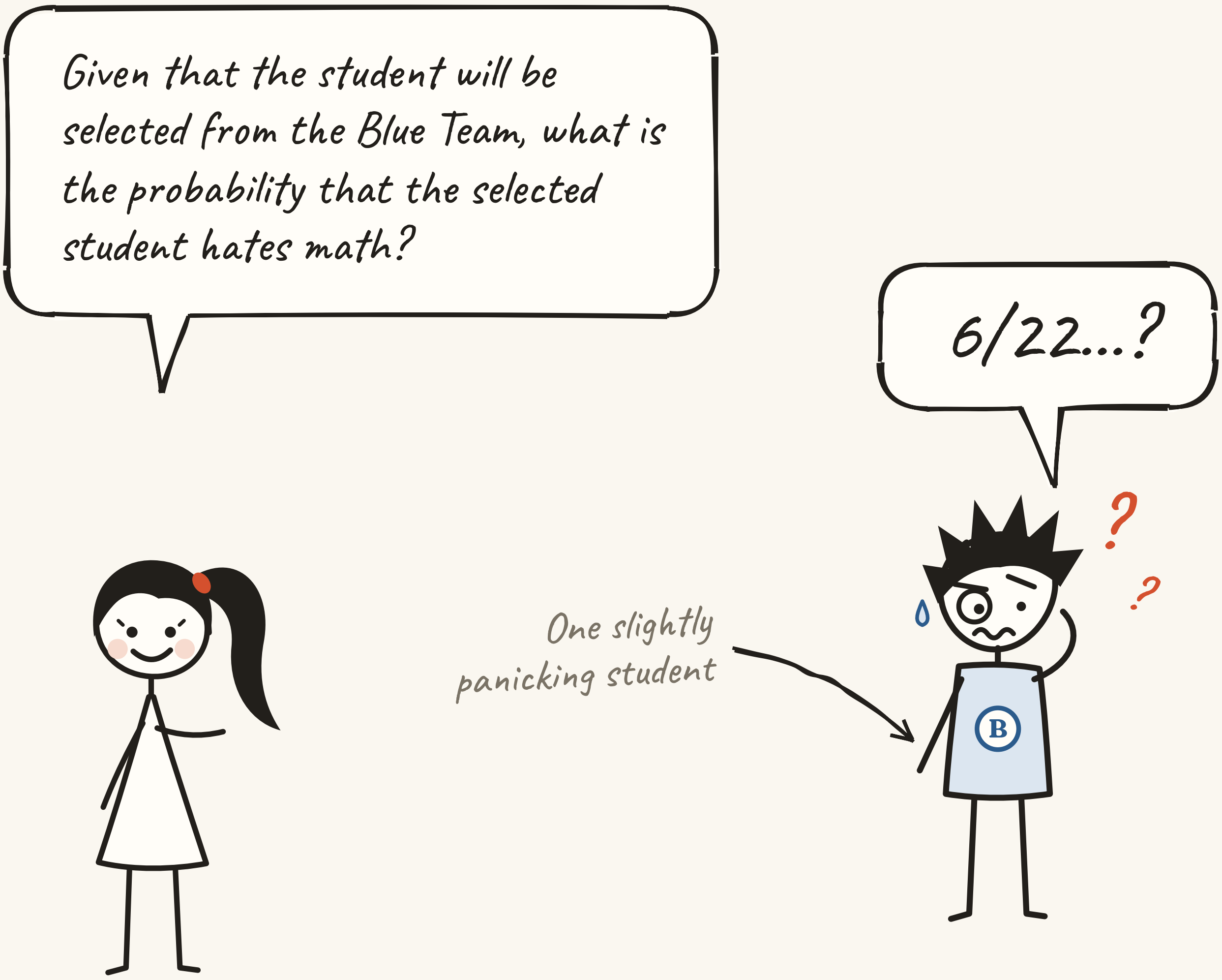
22 students in total



Easy enough. ✓

PART 1 · THE PLOT TWIST

But then, the teacher announces that the next student will be selected *at random from the Blue Team* and asks the class again:



Do you agree with him?

Think about it for a moment...

PART 2 · LET'S EXPLAIN

Let's explain.

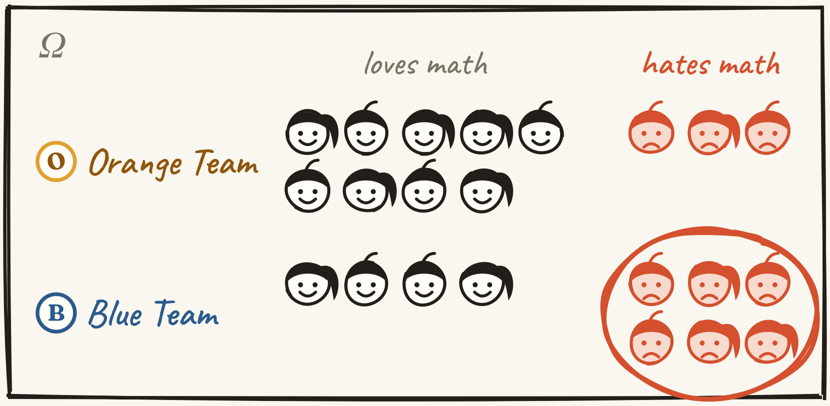
FIRST, SOME NAMES

O	B	H	L
"Orange Team"	*"Blue Team"*	*"hates math"*	*"loves math"*

The teacher's first question can be translated as:

$$P(B \cap H) = \frac{6}{22} \approx 27\%$$

And that makes sense: approximately 27% of *all* students belong to the Blue Team and hate math.



6 Blue Team students who hate math, out of all 22 students

But something changes once we know the next student will come *from the Blue Team...*

PART 2 · LET'S EXPLAIN

Our sample space *shrinks.*

Outside our new sample space

	LOVE MATH	HATE MATH	TOTAL
~~O Orange Team~~	~~9~~	~~3~~	~~12~~
B *Blue Team*	**4**	**6**	**10**
Total	13	9	22

We are no longer considering all 22 students. We already know that the selected student will come from the Blue Team, so students in the Orange Team are outside our restricted sample space.

Our new sample space contains only the **10 students in the Blue Team**.

PART 2 · LET'S EXPLAIN

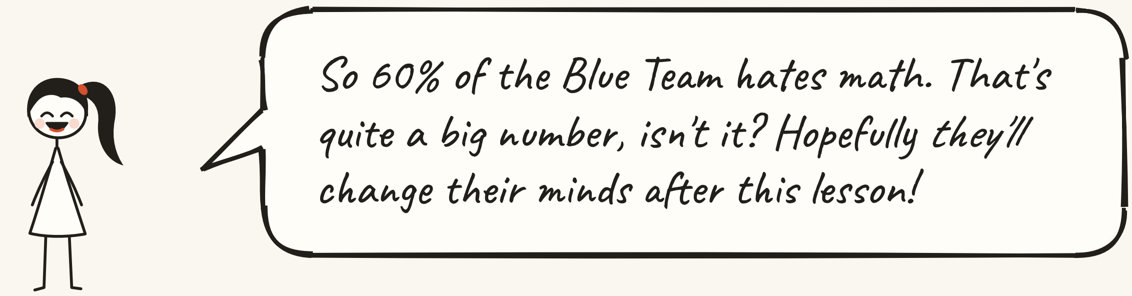
So we can think about the teacher's second question as:

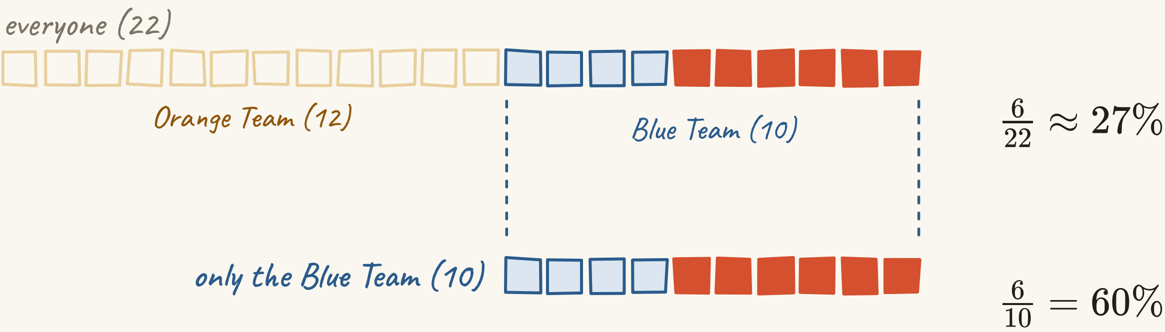
"What percentage of the Blue Team hates math?"

B Blue Team	4	6	10
	LOVE MATH	HATE MATH	TOTAL

There are 10 students in the Blue Team, and 6 of them hate math.

$$\frac{6}{10} = 0.6 = 60\%$$





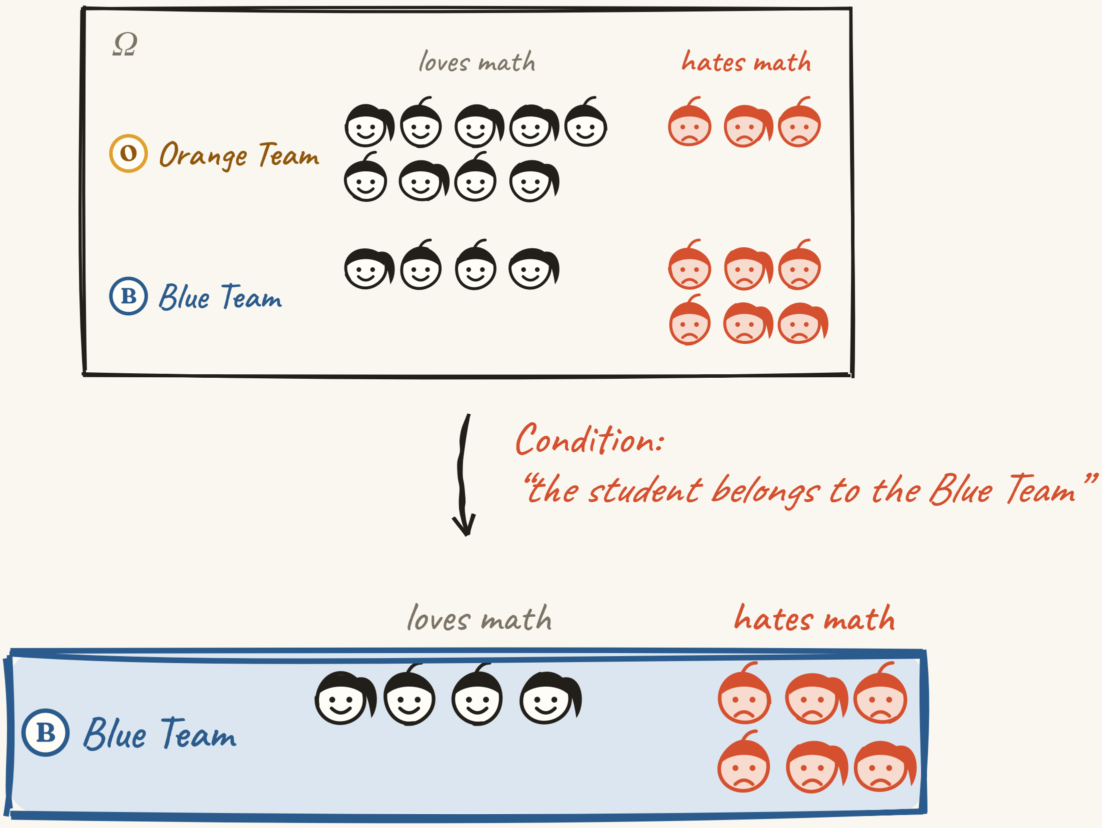
Quite different from 27%, isn't it?

PART 2 · LET'S EXPLAIN

This is what we call a

CONDITIONAL PROBABILITY.

Instead of considering the *entire* sample space, we impose a **condition** that restricts it.



22 possible students → 10 possible Blue Team students.
The Orange Team has not disappeared; its members are simply outside our new sample space.

PART 3 · THE NOTATION

How do we write this?

$$P(H \mid B)$$

"hates math"
what we are asking about

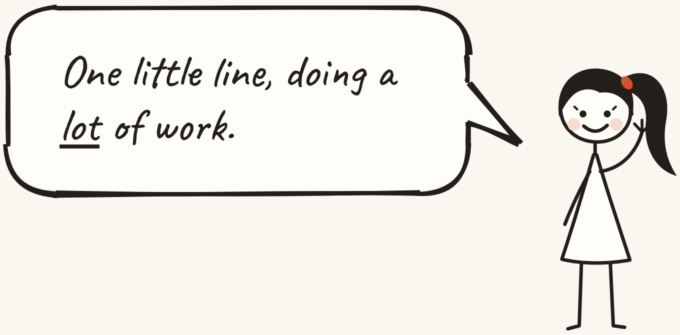
"given"

"belongs to the Blue Team"
what we already know

We read this as:

"The probability of H, given B."

> The symbol | can basically be read as:
>
> *"given that we already know..."*



PART 3 · THE NOTATION

So,

$$P(H \mid B)$$

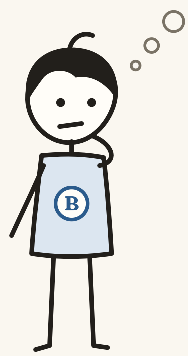
means:

"What is the probability that a student hates math, given that we already know that the student belongs to the Blue Team?"

And from our class, we already know the answer:

$$P(H \mid B) = \frac{6}{10} = 0.6$$

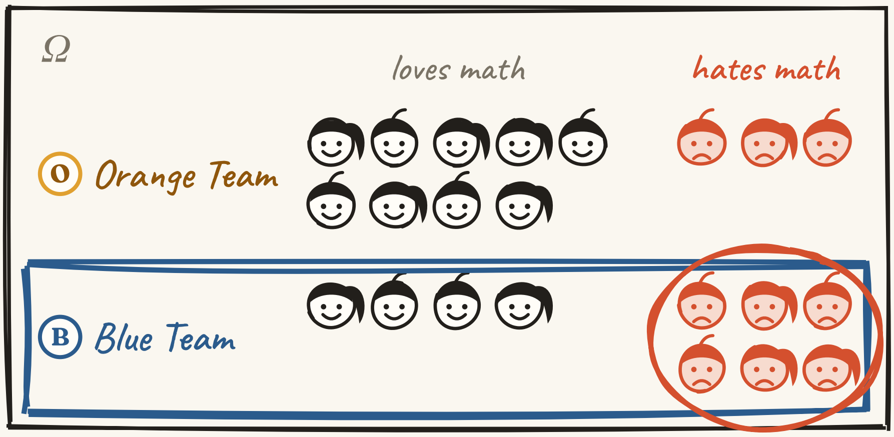
But where does the actual *formula* for conditional probability come from?



PART 4 · LET'S DERIVE IT

Let's derive it.

Start with the whole class. Before we were given any additional information, our sample space contained **22 students**.



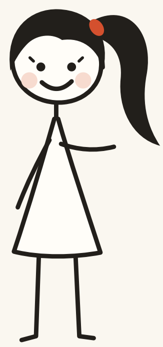
$$P(H \cap B) = \frac{6}{22}$$

The probability that a randomly selected student belongs to the Blue Team *and* hates math is $\frac{6}{22}$.

$$P(B) = \frac{10}{22}$$

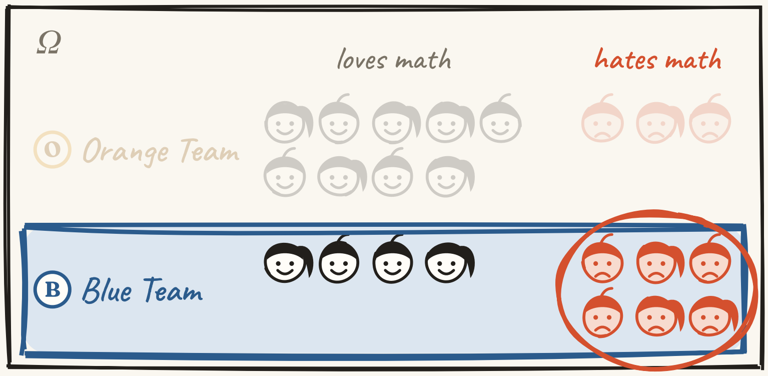
The probability that a randomly selected student belongs to the Blue Team is $\frac{10}{22}$.

Psst: $H \cap B$ and $B \cap H$ are the same event. "Blue Team student who hates math" = "hates math and belongs to the Blue Team."



PART 4 · LET'S DERIVE IT

Now we impose the condition B.



B is our new world.

Everything outside it fades away.

We want to know:

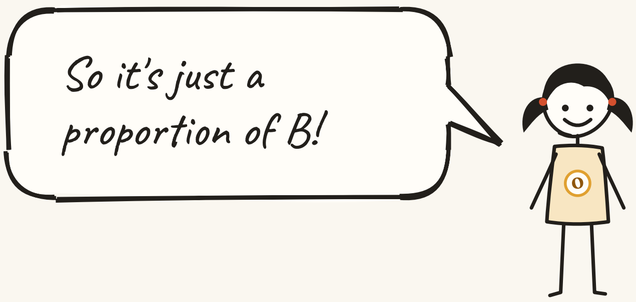
Among the students in the Blue Team, what proportion also belongs to H?

In other words, we want to measure the event $H \cap B$ as a **proportion** of B.

$$\boxed{P(H \mid B) = \frac{P(H \cap B)}{P(B)}}$$

The part of B that also belongs to H

All of B



PART 4 · LET'S DERIVE IT

Now substitute our numbers.

$$\begin{aligned} P(H \mid B) &= \frac{P(H \cap B)}{P(B)} \\ &= \frac{6/22}{10/22} \\ &= \frac{6/\cancel{22}}{10/\cancel{22}} \\ &= \frac{6}{10} \\ &= 0.6 \end{aligned}$$

The factors of 22 cancel because the full class is no longer our reference population. Only the Blue Team remains in the restricted sample space.

And there it is.

PART 5 · THE GENERAL FORMULA

Replace H with any event A while keeping B as the conditioning event. This gives the general definition:

$$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{provided that } P(B) \neq 0}$$

(We can't condition on something impossible. Dividing by zero is frowned upon.)

A NOTE ON RIGOR

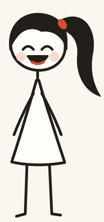
Strictly speaking, this is not a proof of the formula. Conditional probability is **defined** in this way. The classroom example shows why the definition is natural: it measures $A \cap B$ as a proportion of the restricted sample space B.

The important bit.

Don't *just* memorize the formula.

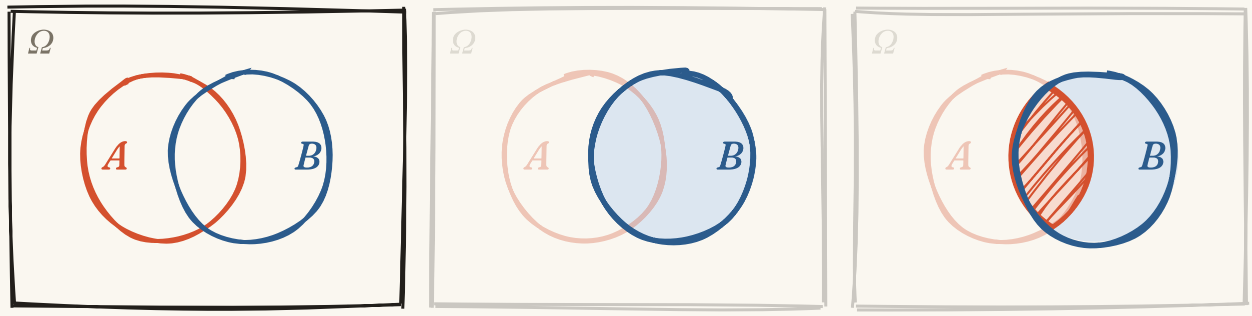
When you see $P(A \mid B)$, think:

“I already know B happened.
So B is my new world.”



PART 5 · THE IMPORTANT BIT

Here's what that looks like.



1. All possible outcomes

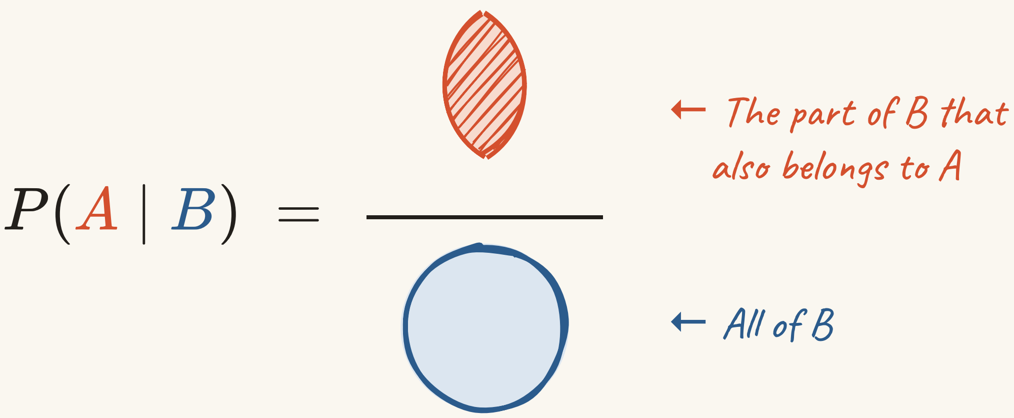
2. B occurred, so outcomes outside B fade

3. How much of B also belongs to A?

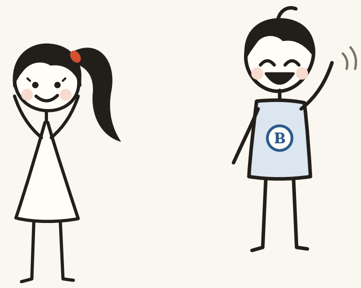
Your sample space has been restricted to B. Now ask:

“What proportion of this new sample space also belongs to A?”

THE SAME THING, IN PICTURES



That's conditional *probability*.



PART 6 · YOUR TURN

Your turn.

Same class, same table.
Grab a pencil.

	LOVE	HATE	TOTAL
Ⓞ *Orange Team*	9	3	12
Ⓑ *Blue Team*	4	6	10
Total	13	9	22

1 Say it in symbols

We select a student at random. Write each probability using O, B, L, H and $\cap$ or $|$.

a. An Orange Team student who loves math.
b. A student who loves math, given that the student belongs to the Orange Team.
c. An Orange Team student among those who hate math.
d. A Blue Team student.

2 The flip trap

Find $P(H \mid B)$ and $P(B \mid H)$. Are they equal? Explain the difference in plain language.

(Hint: What is your new sample space in each case?)

3 Now crunch the numbers

Use the table to find the four probabilities from Exercise 1 as fractions.

a. ---------- b. ---------- c. ---------- d. ----------

4 The sneaky die

Your friend rolls a fair six-sided die and hides the result. She tells you, *"It is even."* What is $P(6 \mid \text{even})$?

Then she adds, *"It is greater than 3."* Given both pieces of information, what is the probability that the result is 6?

5 True or false?

a. Knowing something can only make a probability *smaller*.
b. $P(B \mid B) = 1$
c. $P(A \mid B) = P(B \mid A)$, always.

6 Be the teacher

Invent a question about the class whose answer is exactly $\frac{4}{13}$.

Then try it on a friend and watch them panic.

PART 6 · ANSWER KEY

Answer key.

No peeking unless you've tried first. (We'll know.)

1 Say it in symbols

a. $P(O \cap L)$ **b.** $P(L \mid O)$
c. $P(O \mid H)$ **d.** $P(B)$

Phrases like "given that..." and "among those who..." describe what we already know, so that event goes after the bar. In (a), both things must be true at once, so we use $\cap$.

2 The flip trap

$P(H \mid B) = \frac{6}{10} = 60\%$: among the 10 Blue Team students, 6 hate math.

$P(B \mid H) = \frac{6}{9} = \frac{2}{3} \approx 66.7\%$: among the 9 students who hate math, 6 are on the Blue Team.

Not equal. The event after the bar is the new sample space, so swapping the events changes the denominator (10 vs. 9).

3 Now crunch the numbers

a. $P(O \cap L) = \frac{9}{22}$ *(9 of all 22 students)*

b. $P(L \mid O) = \frac{9}{12} = \frac{3}{4}$ *(9 of the 12 Orange Team students)*

c. $P(O \mid H) = \frac{3}{9} = \frac{1}{3}$ *(3 of the 9 students who hate math)*

d. $P(B) = \frac{10}{22} = \frac{5}{11}$ *(10 of all 22 students)*

4 The sneaky die

$P(6 \mid \text{even}) = \frac{1}{3}$: the new sample space is {2, 4, 6}, and one of those three outcomes is 6.

$P(6 \mid \text{even and greater than } 3) = \frac{1}{2}$: the sample space shrinks again, to {4, 6}.

More information, smaller sample space.

5 True or false?

a. False. $P(H) = \frac{9}{22} \approx 41\%$, but $P(H \mid B) = 60\%$. Knowing B made H *more* likely.

b. True (as long as $P(B) \neq 0$): $P(B \mid B) = \frac{P(B \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$. If you already know B happened, it's certain.

c. False. In Exercise 2, $P(H \mid B) = 60\%$ but $P(B \mid H) \approx 66.7\%$.

6 Be the teacher

One possible answer: *"The teacher selects a student who loves math. What is the probability that the student belongs to the Blue Team?"*

$P(B \mid L) = \frac{4}{13}$ *because 13 students love math, and 4 of them are on the Blue Team.*